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SHORTER COMMUNICATIONS

CONVECTIVE DIFFUSION IN AXISYMMETRICAL STAGNATION FLOW FOR DETERMINATION OF PROPERTIES OF MEMBRANES FOR REVERSE OSMOSIS DESALINATION OF WATER

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NOMENCLATURE

A,	membrane constant as defined by equation (6);
B,	dimensionless osmotic pressure defined as $\pi_e/\Delta P$;
c,	mass fraction of salt in saline water;
d,	diameter of nozzle;
D,	diffusion coefficient;
f,	dimensionless stream function defined by equation (9);
I,	integral as defined by equation (19);
J,	coefficient as defined by equation (20);
m_w,	mass flux of salt at membrane surface;
m_p,	total mass flux of product water;
ΔP,	pressure drop across membrane;
R,	membrane constant, measure of salt rejection defined as $(1 - c_p/c_w)$;
Sc,	Schmidt number;
u,	velocity in x direction;
$\bar{u}$,	transformed velocity as defined by equation (8);
u_∞,	velocity at jet nozzle;
v,	velocity in y direction;
$\bar{v}$,	transformed velocity as defined by equation (8);
w,	variable proportional to η as defined by equation (18);
x,	distance along surface in direction of mainstream flow;
$\bar{x}$,	transformed x as defined by equation (8);
y,	normal distance from interface towards liquid;
$\bar{y}$,	transformed y as defined by equation (8).

Greek symbols

η,	dimensionless co-ordinate as defined by equation (9);
θ,	dimensionless salt concentration defined as c/c_w;
ν,	kinematic viscosity;
ξ,	transformed co-ordinate as defined by equation (9);
π_e,	osmotic pressure of free stream saline water;
ρ,	density;
ψ,	stream function as defined by equation (9).

Subscripts

e,	value in free stream;
p,	value in product solution;
w,	value at membrane surface;
∞,	value in jet nozzle.

1. INTRODUCTION

IN WATER desalination by reverse osmosis, product water is removed from saline solution by permeation through a semipermeable membrane which rejects the dissolved salts more or less completely. Such membrane characteristics as its salt rejection capability, hydraulic permeability and lifetime have a marked effect on the capital investment and operating cost of reverse osmosis process for desalination [1]. Several reverse osmosis systems, for example one dimensional transient model [2], plane stagnation flow [3] and rotating disk system [4], have been suggested to be used for laboratory determination of membrane properties.

In the present paper the reverse osmosis desalination process in axisymmetrical stagnation flow is examined with reference to the determination of membrane properties. This system has the same advantage as the last two systems mentioned above; that is, incomplete salt rejection may be easily and exactly treated since the salt concentration at the membrane surface is a constant, which is not true for the one dimensional transient model mentioned above. Another advantage of the present system is that the axisymmetrical stagnation flow is easily contrived in the laboratory.

The objective of this paper is twofold, namely: (i) to discuss the process of reverse osmosis in axisymmetrical stagnation flow, both exact and accurate approximate solutions will be presented; and (ii) to describe an application of these solutions to the determination of membrane constants.

A description of a particular axisymmetrical stagnation flow with reference to reverse osmosis system now follows, in order that a concrete physical situation and definite theoretical question may be held in mind during the subsequent analysis. Figure 1 illustrates a perpendicular jet. A jet of saline water of known salt concentration flows towards a perpendicular surface, which deflects the stream radially outward. We may write for the central portion of the jet [5–7],

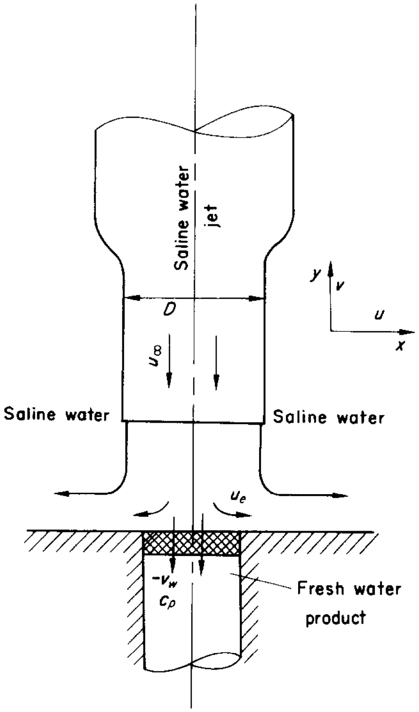

Fig. 1. Axisymmetrical stagnation flow resulting from impingement of a jet on a perpendicular surface.

$$u_e/x = \mathrm{d}u_e/\mathrm{d}x = F(u_\infty, d). \tag{1}$$

In this flow, u_e is proportional to x, which is the condition for the stagnation flow. The proportionality constant u_e/x is of course a function of velocity and diameter of the jet, which may be calibrated once and for all.

Application of the perpendicular jet for laboratory testing of stagnation point ablation and transpiration cooling was first suggested by Professor Spalding and has since been successfully carried out in several laboratories [6, 7]. For laboratory determination of membrane properties, a specimen of the membrane may be placed at the central portion of the surface. If steady conditions prevail, membrane constants, as will be shown below, may be deduced from water production and qualities.

2. ANALYSIS

Basic equations

When rectangular co-ordinates and usual assumptions of uniform physical properties and density are used, the boundary-layer equations for the system described above may be written as:

$$\partial ux/\partial x + \partial vx/\partial y = 0, \tag{2}$$

$$u\partial u/\partial x + v\partial u/\partial y = u_e\,\mathrm{d}u_e/\mathrm{d}x + \nu\partial^2 u/\partial y^2, \tag{3}$$

and

$$u\partial c/\partial x + v\partial c/\partial y = D\partial^2 c/\partial^2 y, \tag{4}$$

with the boundary conditions,

$$\begin{aligned} y = 0: &\quad u = 0, \quad v = v_w, \quad c = c_w; \\ y = \infty: &\quad u = u_e, \quad c = c_e. \end{aligned} \tag{5}$$

For desalination process, flow through the membrane may be described by:

$$-v_w = A\Delta P\left[1 - RB(c_w/c_e)\right]/(1 - c_p). \tag{6}$$

The mass flux of salt at membrane surface is given by:

$$m_w = c_w\rho v_w - \rho D(\partial c/\partial y)_w$$

so that we can write:

$$(D/c_w)(\partial c/\partial y)_w = Rv_w. \tag{7}$$

In the above equations (6) and (7), A is the membrane constant which measures the water production and R is another constant which measures the salt rejection.

The following transformations are introduced:

$$\bar{x} \equiv \int (x/d)^2\,\mathrm{d}x, \bar{y} \equiv xy/D, \bar{u} \equiv u, \bar{v} \equiv (d/x)(v + yu/x); \tag{8}$$

and

$$\begin{aligned} &\bar{u} \equiv \partial\psi/\partial\bar{x}, \bar{v} \equiv -\partial\psi/\partial\bar{y}, \\ &\eta \equiv \bar{y}(2u_e/3\nu\bar{x})^{\frac{1}{2}}, f(\eta) \equiv (\psi/u_e)(2u_e/3\nu\bar{x})^{\frac{1}{2}}, \\ &\xi \equiv \bar{x}, \theta(\eta) \equiv c/c_w. \end{aligned} \tag{9}$$

where the transformation defined by (8) is the well-known Mangler transformation ([8], p. 426), and that defined by (9) is the usual similarity transformation for 2-dimensional flow ([8], p. 235).

For the axisymmetrical stagnation flow u_e/x is a constant which is a function of d and u_∞ but independent of x. Equations (2)–(5) reduce to:

$$f''' + ff'' + \tfrac{1}{2}\left[1 - (f')^2\right] = 0, \tag{10}$$

and

$$\theta'' + Scf\theta' = 0, \tag{11}$$

with the boundary conditions:

$$\eta = 0: f = f(0), f' = 0, \theta = 1; \eta = \infty: f' = 1, \theta = \theta_e. \tag{12}$$

In the above equations, primes denote differentiation with respect to η.

Equation (7) provides another relation to be satisfied at $\eta = 0$:

$$\theta'(0) = -RScf(0). \tag{13}$$

Twice integrating equation (11) and letting $\eta \to \infty$, we obtain:

$$\theta_e = 1 - RScf(0) \int_0^\infty \exp\left(-Sc \int_0^\eta f \, d\eta\right) d\eta$$

$$= 1 - R(1 - \theta_e|_{R=1}). \tag{14}$$

Equation (14) indicates that the solution θ_e for arbitrary R can be obtained from the solution for $R = 1$. Furthermore, equation (14) can be used to develop a relation for R [3]:

$$R = [1 - (c_p/c_e)]/[1 - (c_p/c_e)(1 - \theta_e|_{R=1})] \tag{15}$$

Numerical solution for similar velocity equation

It may be shown that a major range of polarization of interest for reverse osmosis corresponds to $0 < f(0) < 0{\cdot}02$ [3]. Velocity equation (10) with appropriate boundary conditions have been solved for $f(0)$ from 0 to 0·04 by a Runge–Kutta process. The techniques employed are too well known to warrant detailed discussion ([8], p. 67]. Values of $f''(0)$ for the case of $f(0)$ from 0 to 0·04 are shown in Table 1.

Table 1. *Axisymmetrical stagnation flow* $f(0)$ *and* $f''(0)$; *and* $\theta_e|_{R=1, Sc=560}$

$f(0)$	$f''(0)$	$\theta_e\|_{R=1, Sc=560}$	
		Exact	Approximate
~0	0·9278	1·0000	1·0000
0·005	0·9307	0·5775	0·5786
0·010	0·9337	0·3481	0·3485
0·015	0·9367	0·2183	0·2178
0·020	0·9397	0·1420	0·1408
0·040	0·9518	0·0318	0·0315

Solution for concentration equation

Integration of equation (14) may be carried out directly by a numerical procedure. However, an accurate approximate procedure for large values of Sc is to approximate f in the concentration equation by:

$$f = f(0) + f''(0)\eta^2/2. \tag{16}$$

On insertion of this expression for f in the integral, equation (14), we obtain the relation,

$$1 - \theta_e = RScf(0) \int_0^\infty \exp\{-Sc[f(0)\eta + f''(0)\eta^3/6]\} \, d\eta. \tag{17}$$

After introduction of a new variable of integration, w, defined by:

$$w = [Scf''(0)/6]^{\frac{1}{3}} \eta. \tag{18}$$

Equation (17) reduces to

$$1 - \theta_e = RJ \int_0^\infty \exp(-Jw - w^3) \, dw$$

$$= RI(J), \text{ (say)}, \tag{19}$$

where

$$J \equiv Sc^{\frac{2}{3}} f(0) [6/f''(0)]^{\frac{1}{3}}. \tag{20}$$

Equation (20) shows that J can be ascribed a particular number whenever $f(0)$, $f''(0)$ and Sc are known. Now the relation between $f(0)$ and $f''(0)$ is known from Table 1. So $I(J)$ is a function of J, it now follows that it too can be evaluated if $f(0)$, $f''(0)$ and Sc are prescribed.

For convenience of user of the theory, values of $I(J)$ over a range of J of practical interest for desalination is presented in Table 2.

Table 2. *Values of* J, I *and* $\theta_e|_{R=1}$

J	I	$\theta_e\|_{R=1}$
0	0	1·0000
0·1	0·0849	0·9151
0·2	0·1618	0·8382
0·3	0·2314	0·7686
0·4	0·2945	0·7055
0·5	0·3517	0·6483
1·0	0·5689	0·4311
1·5	0·7062	0·2938
2·0	0·7950	0·2050
2·5	0·8538	0·1462
3·0	0·8935	0·1065
4·0	0·9402	0·0598
5·0	0·9640	0·0360

For $R = 1$, θ_e by the present approximation reduces to $(1 - I)$. Values of the approximate θ_e for $R = 1$, $Sc = 560$, and $f(0)$ from 0 to 0·04 are shown in Table 1. Also shown in Table 1 are values of exact θ_e, for the same condition, calculated by numerical integration of equation (14). The present approximate solution can be seen to be very accurate, within 1 per cent.

3. DETERMINATION OF MEMBRANE PROPERTIES

In the laboratory determination of membrane properties, the problem may be specified as follows:

The relationship between u_e/x vs. u_∞ for a given apparatus has been calibrated once and for all. In each test, the brine properties (ν and Sc) and the jet conditions (c_∞ and u_∞) are specified; and water production rate (m_p) and quality (c_p), and the average pressure drop across membrane are measured. It is required to reduce the values of A, R and c_w.

The procedure of solving this problem is as follows:

(i) The values of c_e, u_e/x and v_w are deduced from the data c_∞, u_∞ and m_p in a straightforward manner.

(ii) The value of $f(0)$ is calculated by the equation $f(0) = -v_w[x/2\nu u_e)]^{\frac{1}{2}}$; $f''(0)$ is read from Table 1; J is calculated from equation (20); and I is read from Table 2.

(iii) The membrane constant for salt rejection R is calculated by use of equations (15) and (19), i.e.

$$R = (1 - c_p/c_e)/(1 - c_p I(J)/c_e). \qquad (21)$$

(iv) Salt concentration at membrane surface is calculated by the equation,

$$c_w = c_e/\theta_e, \qquad (22)$$

where θ_e is obtained from equations (14) and (19), i.e.

$$\theta_e = 1 - RI(J). \qquad (23)$$

(v) Membrane constant A is calculated by equation (6),

$$A = v_w(1 - c_p)/[\Delta P(1 - RB/\theta_e)], \qquad (24)$$

where θ_e is given by equation (23) and B is the dimensionless osmotic pressure which is assumed to be known.

From the above procedure corresponding values of R, A and c_w for a membrane are deduced from test data. The difficult task of measuring c_w is avoided; or should c_w be measured, the above deduced value of c_w would provide a double checking.

4. CONCLUSIONS

The reverse osmosis process in axisymmetrical stagnation flow has been studied with specific reference to determination of membrane properties. Perpendicular jets have been used for laboratory testing of stagnation point ablation and transpiration cooling. They also appear capable of providing a simple means for testing of membranes.

The axisymmetrical stagnation point boundary layer equations are transformed to the ordinary velocity and concentration differential equations. The velocity equation is solved numerically by Runge–Kutta method. The concentration equation is solved by a very accurate approximate procedure for the range of Schmidt number Sc and suction parameter $f(0)$ of practical interest for desalination. At $Sc = 560$ (common salt at room temperature), approximate solution is compared with exact numerical solution; the agreement is within 1 per cent for the whole range of interest.

Results of the present analysis are presented in Tables 1 and 2 for the convenience of the users of the theory. Use of the present results to facilitate empirical determination of membrane properties is discussed in detail.

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